Journal of Engineering Physics and Thermophysics, Vol. 65, No. 6, 1993

GENERALIZED HEAT CONDUCTION IN COATED BODIES THAT ACCOUNTS FOR THE COATING CURVATURE

Yu. M. Kolyano* and M. Ye. Khomyakevich

UDC 539.3

Generalized conditions for heat transfer of a body through a thin coating are formulated with account for the coating curvature.

Coated materials are used extensively as structural elements in various areas of modern technology where new high-strength materials capable of operating in corrosive media at elevated temperatures and pressures are needed. It turns out that some materials that do not have these capabilities start to meet these demands after being covered with a thin layer of special coating.

Generalized boundary conditions of heat transfer for bodies with thin coatings and generalized conditions of nonideal thermal contact between dissimilar solids were formulated in [1-4] without account for the curvature of the middle surface of an intermediate layer.

Let us formulate the generalized boundary conditions of heat transfer for bodies with thin coatings with account for the curvature of the middle surface of the coating.

Suppose a body is covered with a thin layer of a different material. There are generalized conditions of ideal thermal contact between the body and the coating, whereas heat transfer on the coating-medium boundary follows the generalized Newton law.

In this case, to determine the temperature field in the considered piecewise homogeneous body referred to the system of curvilinear orthogonal coordinates (α, β, γ), we have the equation of generalized heat conduction [5, 6]

$$\frac{\lambda_i}{H_1 H_2 H_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{H_2 H_3}{H_1}\frac{\partial t_i}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{H_1 H_3}{H_2}\frac{\partial t_i}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{H_1 H_2}{H_3}\frac{\partial t_i}{\partial\gamma}\right)\right]=l_i\left(c_i t_i - w_i\right), \tag{1}$$

generalized conditions of ideal contact [6]

$$t_0 = t_1\,,\quad \frac{\lambda_0}{\tau_r^{(0)}}\int_0^{\tau}\varphi_0(\xi\,,\tau)\,d\xi=\frac{\lambda_1}{\tau_r^{(1)}}\int_0^{\tau}\varphi_1(\xi\,,\tau)\,d\xi \quad \text{at} \quad \gamma=-\delta\,, \tag{2}$$

the generalized boundary condition

$$\frac{\partial t_0}{\partial\gamma}+\frac{\alpha_0}{\lambda_0}l_0\left(t_0-t_{\mathrm{m}}\right)=0 \quad \text{at} \quad \gamma=+\delta \tag{3}$$

and the initial conditions

$$t_0=t_0^{(0)}\,,\quad \dot{t}_0=0\,,\quad t_1=t_1^{(0)}\,,\quad \dot{t}_1=0 \quad \text{at} \quad \tau=0\,, \tag{4}$$

* Deceased.

Institute for the Applied Problems of Mechanics and Mathematics, L'vov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 65, No. 6, pp. 745-749, December, 1993. Original article submitted October 10, 1991.

where

$$l_i = 1 + \tau_r^{(i)} \frac{\partial}{\partial \tau}, \quad \varphi_i(\xi, \tau) = \exp\left(\frac{\xi - \tau}{\tau_r^{(i)}}\right) \frac{\partial t_i}{\partial \gamma}, \quad t_i = \frac{\partial t_i}{\partial \tau}, \quad i = 0, 1.$$

For determining the temperature field of the coating, we have a heat conduction equation that is referred to the mixed system of coordinates (α, β, γ) and that after certain simplifications [6] takes the form

$$\frac{\partial^2 t_0}{\partial \gamma^2} + 2k \frac{\partial t_0}{\partial \gamma} + p^2 t_0 = - l_0 \frac{w_0}{\lambda_0}, \tag{5}$$

where

$$p^2 = \Delta - \frac{c_0}{\lambda_0} l_0 \frac{\partial}{\partial \tau}, \quad \Delta = \frac{1}{AB}\left[\frac{\partial}{\partial \alpha}\left(\frac{B}{A}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{A}{B}\frac{\partial}{\partial \beta}\right)\right].$$

The temperature of the coating satisfies relations (2) and conditions (3) and (4).

By averaging Eq. (5) in conformity with the integral characteristics of the temperature

$$T = \frac{1}{2\delta} \int_{-\delta}^{\delta} t_0 d\gamma, \quad T^* = \frac{3}{2\delta^2} \int_{-\delta}^{\delta} \gamma t_0 d\gamma, \tag{6}$$

we correspondingly obtain

$$\Lambda_0 p^2 T + \lambda_0 \left[\left(\frac{\partial t_0}{\partial \gamma}\right)^+ - \left(\frac{\partial t_0}{\partial \gamma}\right)^-\right] + 2k\lambda_0 (\overset{+}{t_0} - t_1) = - l_0 W_0,$$

$$\Lambda_0 p^2 T^* + 3\lambda_0 \left[\left(\frac{\partial t_0}{\partial \gamma}\right)^+ + \left(\frac{\partial t_0}{\partial \gamma}\right)^-\right] - \frac{6}{r_0}(\overset{+}{t_0} - t_1) +$$

$$+ 6k\lambda_0 (\overset{+}{t_0} + t_1) - 12k\lambda_0 T = - l_0 W_0^*, \tag{7}$$

where

$$\Lambda_0 = 2\lambda_0 \delta; \quad r_0 = \frac{2\delta}{\lambda_0}; \quad \overset{+}{t_0} = t_0|_{\gamma=+\delta}; \quad t_1 = t_0|_{\gamma=-\delta};$$

$$\left(\frac{\partial t_0}{\partial \gamma}\right)^{\pm} = \left(\frac{\partial t_0}{\partial \gamma}\right)_{\gamma=\pm\delta}; \quad W_0 = \int_{-\delta}^{\delta} w_0 d\gamma; \quad W_0^* = \frac{3}{\delta}\int_{-\delta}^{\delta} w_0 \gamma d\gamma.$$

Using an operational method, we write the general solution of Eq. (5) in the form

$$t_0 = \exp(-k\gamma)\left[\frac{E \cos \bar{p}\gamma + F \sin \bar{p}\gamma}{\left(-k + \frac{\alpha_0 l_0}{\lambda_0}\right)\sin 2\bar{p}\delta + \bar{p}\cos 2\bar{p}\delta} + \frac{l_0 Q_0}{\lambda_0 \bar{p}^2}\right], \tag{8}$$

where

$$E = (\bar{p}\cos \bar{p}\delta - k \sin \bar{p}\delta)\, t_1 \exp(-k\delta) +$$

$$+ \frac{\alpha_0 l_0}{\lambda_0} (\exp(k\delta)\, t_m + \exp(-k\delta)\, t_1) \sin \bar{p}\delta - \frac{l_0 q_{01}}{\lambda_0 \bar{p}^2}; \quad F = (k \cos \bar{p}\delta +$$

$$+ \bar{p} \sin \bar{p}\delta)\, t_1 \exp(-k\delta) + \frac{\alpha_0 l_0}{\lambda_0} (\exp(k\delta)\, t_m - \exp(-k\delta)\, t_1) \cos \bar{p}\delta - \frac{l_0 q_{02}}{\lambda_0 \bar{p}^2};$$

$$q_{01} = \bar{p} Q_0^- \cos \bar{p}\delta + \left(\frac{\partial Q_0}{\partial \gamma} \right)^+ \sin \bar{p}\delta + \frac{\alpha_0 l_0}{\lambda_0} (Q_0^+ + Q_0^-) \sin \bar{p}\delta;$$

$$q_{02} = \bar{p} Q_0^- \sin \bar{p}\delta + \left(\frac{\partial Q_0}{\partial \gamma} \right)^+ \cos \bar{p}\delta + \frac{\alpha_0 l_0}{\lambda_0} (Q_0^+ - Q_0^-) \cos \bar{p}\delta;$$

$$Q_0 = \bar{p} \int_0^{\nu} \sin \bar{p} (\xi - \gamma)\, w_0 (\xi, \tau)\, d\xi; \quad \left(\frac{\partial Q_0}{\partial \gamma} \right)^{\pm} = \left(\frac{\partial Q_0}{\partial \gamma} \right)_{\gamma = \pm\delta};$$

$$Q_0^{\pm} = Q_0 |_{\gamma = \pm\delta}; \quad \bar{p}^2 = p^2 - k^2.$$

Substituting Eq. (8) into Eq. (6), we find

$$T = \frac{E (k \cos \bar{p}\delta \,\mathrm{sh}\, k\delta + \bar{p} \sin \bar{p}\delta \,\mathrm{ch}\, k\delta) - F (k \sin \bar{p}\delta \,\mathrm{ch}\, k\delta - \bar{p} \cos \bar{p}\delta \,\mathrm{sh}\, k\delta)}{\delta (k^2 + \bar{p}^2) \left[\left(-k + \frac{\alpha_0 l_0}{\lambda_0} \right) \sin 2\bar{p}\delta + \bar{p} \cos 2\bar{p}\delta \right]} + \frac{l_0 \tilde{Q}_0}{\lambda_0 \bar{p}^2}, \tag{9}$$

where

$$T^* = 3 \frac{EE^* - FF^*}{\delta^2 (k^2 + \bar{p}^2) \left[\left(-k + \frac{\alpha_0 l_0}{\lambda_0} \right) \sin 2\bar{p}\delta + \bar{p} \cos 2\bar{p}\delta \right]} + \frac{l_0 \tilde{Q}_0^*}{\lambda_0 \bar{p}^2},$$

$$E^* = -k\delta \cos \bar{p}\delta \,\mathrm{ch}\, k\delta + \frac{k^2 - \bar{p}^2}{k^2 + \bar{p}^2} \cos \bar{p}\delta \,\mathrm{sh}\, k\delta - \bar{p}\delta \,\mathrm{sh}\, k\delta \sin \bar{p}\delta +$$

$$+ \frac{2k\bar{p}}{k^2 + \bar{p}^2} \sin \bar{p}\delta \,\mathrm{ch}\, k\delta;$$

$$F^* = -k\delta \sin \bar{p}\delta \,\mathrm{sh}\, k\delta + \frac{k^2 - \bar{p}^2}{k^2 + \bar{p}^2} \sin \bar{p}\delta \,\mathrm{ch}\, k\delta + \bar{p}\delta \cos \bar{p}\delta \,\mathrm{ch}\, k\delta -$$

$$- \frac{2k\bar{p}}{k^2 + \bar{p}^2} \cos \bar{p}\delta \,\mathrm{sh}\, k\delta;$$

$$\tilde{Q}_0 = = \frac{1}{2\delta} \int_{-\delta}^{\delta} Q_0 d\gamma; \quad \tilde{Q}_0^* = \frac{3}{2\delta^2} \int_{-\delta}^{\delta} \gamma Q_0 d\gamma.$$

We now incorporate into relations (7) the second condition (2) and expressions (9) and in the resulting relation we let δ go to zero, keeping unchanged the thermal conductivity $\Lambda_0 = 2\lambda_0\delta$, the heat capacity $C_0 = 2l_0\delta$, the internal resistance of the coating $r_0 = 2\delta/\lambda_0$, the products $\Lambda_0 r_0$, $C_0 r_0$, and W_0, W_0^*. As a result, we arrive at the following generalized condition of heat transfer through a thin coating:

$$\Lambda_0\Delta\left\{\left(1+\frac{\alpha_0 l_0+2k\lambda_0}{2h}\right)t_1+\frac{\lambda_1}{2h\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial\gamma}+\right.\right.$$

$$\left.\left.+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\cdot\right]\right\}+\alpha_0 l_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-$$

$$-\left(1+\frac{\alpha_0 l_0}{h}\right)\frac{\lambda_1}{\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial\gamma}+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\right]=$$

$$=C_0 l_0\left\{\left(1+\frac{\alpha_0 l_0+2k\lambda_0}{2h}\right)\frac{\partial t_1}{\partial\tau}+\frac{\lambda_1}{2h\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial^2 t_1}{\partial\gamma d\tau}+\right.\right.$$

$$\left.\left.+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)\left(\frac{\partial t_1}{\partial\gamma}-\frac{1}{\tau_r^{(1)}}\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\right)\right]\right\}+l_0\overline{W}_0, \tag{10}$$

where $\overline{W}_0=W_0-\dfrac{W_0^*}{3}$; $r_0=1/h$.

The following are some particular cases of condition (10).

1. Suppose $\tau_r^{(0)}\to 0$, $\tau_r^{(1)}\neq 0$. Then we obtain the condition:

$$\Lambda_0\Delta\left[\left(1+\frac{\alpha_0+2k\lambda_0}{2h}\right)t_1+\frac{\lambda_1}{2h\tau_r^{(1)}}\times\right.$$

$$\left.\times\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\cdot\right]+\alpha_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-$$

$$-\left(1+\frac{\alpha_0}{h}\right)\frac{\lambda_1}{\tau_r^{(1)}}\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi=$$

$$=C_0\left[\left(1+\frac{\alpha_0+2k\lambda_0}{2h}\right)\frac{\partial t_1}{\partial\gamma}+\right.$$

$$\left.+\frac{\lambda_1}{2h\tau_r^{(1)}}\left(\frac{\partial t_1}{\partial\gamma}-\frac{1}{\tau_r^{(1)}}\int_0^\tau\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\right)\right]+\overline{W}_0. \tag{11}$$

2. But if $\tau_r^{(0)}\neq 0$, $\tau_r^{(1)}\to 0$, we have

$$\Lambda_0\Delta\left[\left(1+\frac{\alpha_0 l_0+2k\lambda_0}{2h}\right)t_1+\frac{\lambda_1 l_0}{2h}\frac{\partial t_1}{\partial\gamma}\right]+$$

$$+\alpha_0 l_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-\left(1+\frac{\alpha_0 l_0}{h}\right)\lambda_1 l_0\frac{\partial t_1}{\partial\gamma}=$$

$$=C_0 l_0\left[\left(1+\frac{\alpha_0+2k\lambda_0}{2h}\right)\frac{\partial t_1}{\partial\tau}+\frac{\lambda_1 l_0}{2h}\frac{\partial^2 t_1}{\partial\gamma\partial\tau}\right]+l_0\overline{W}_0. \tag{12}$$

3. For the classical case ($\tau_r^{(0)}\to 0$, $\tau_r^{(1)}\to 0$), we obtain the relationship

$$\Lambda_0\Delta\left[\left(1+\frac{\alpha_0+2k\lambda_0}{2h}\right)t_1+\frac{\lambda_1}{2h}\frac{\partial t_1}{\partial\gamma}\right]+\alpha_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-$$

$$-\lambda_1\left(1+\frac{\alpha_0}{h}\right)\frac{\partial t_1}{\partial\gamma}=C_0\left[\left(1+\frac{\alpha_0+2k\lambda_0}{2h}\right)\frac{\partial t_1}{\partial\tau}+\frac{\lambda_1}{2h}\frac{\partial^2 t_1}{\partial\gamma\partial\tau}\right]+\overline{W}_0, \tag{13}$$

which is given in [7] for $k = 0$ and $\omega_0 = 0$.

4. For metals ($\tau_r^{(0)} = \tau_r^{(1)}$), we have

$$\Lambda_0\Delta\left[\left(1+\frac{\alpha_0 l_0+2k\lambda_0}{2h}\right)t_1+\frac{\lambda_1}{2h}\frac{\partial t_1}{\partial\gamma}\right]+$$

$$+\alpha_0 l_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-\lambda_1\left(1+\frac{\alpha_0 l_0}{h}\right)\frac{\partial t_1}{\partial\gamma}=$$

$$=C_0 l_0\left[\left(1+\frac{\alpha_0 l_0+2k\lambda_0}{2h}\right)\frac{\partial t_1}{\partial\tau}+\frac{\lambda_1}{2h}\frac{\partial^2 t_1}{\partial\gamma\partial\tau}\right]+l_0\overline{W}_0. \tag{14}$$

If we neglect the products $\Lambda_0 r_0$ and $C_0 r_0$ in conditions (11)-(14), we obtain simpler conditions. In particular, relations (10) and (13) can be represented as

$$\Lambda_0\Delta t_1+\alpha_0 l_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-\left(1+\frac{\alpha_0 l_0}{h}\right)\frac{\lambda_1}{\tau_r^{(1)}}\left[\tau_r^{(0)}\frac{\partial t_1}{\partial\gamma}+\right.$$

$$\left.+\left(1-\frac{\tau_r^{(0)}}{\tau_r^{(1)}}\right)\int_0^{\tau}\exp\left(\frac{\xi-\tau}{\tau_r^{(1)}}\right)\frac{\partial t_1}{\partial\gamma}d\xi\right]=C_0 l_0\frac{\partial t_1}{\partial\tau}+l_0\overline{W}_0, \tag{15}$$

$$\Lambda_0\Delta t_1+\alpha_0\left(1+\frac{k\lambda_0}{h}\right)(t_m-t_1)-\lambda_1\left(1+\frac{\alpha_0}{h}\right)\frac{\partial t_1}{\partial\gamma}=C_0\frac{\partial t_1}{\partial\gamma}+\overline{W}_0. \tag{16}$$

We have formulated the complicated boundary conditions (10)-(16) for heat transfer through a thin coating that is characterized by the above thermophysical parameters. We have managed to exclude from consideration the region occupied by the coating. Its effect is taken into account by complicated boundary conditions involving first- and second-order time and space derivatives.

NOTATION

t_i, ω_i, $\tau_r^{(i)}$ ($i = 0, 1$), temperature, density of heat sources, and time of the heat flux relaxation of the coating and the main material; λ_i, c_i, thermal conductivity and volume heat capacity of the coating and the main material, respectively; α_0, coefficient of heat transfer from the surface $\gamma = +\delta$; t_m, temperature of the medium flowing past this surface; τ, time; h, thermal conductivity of the coating; k, curvature of the midddle surface of the coating; 2δ, coating thickness; H_1, H_2, H_3, Lamé coefficients; A, B, coefficients of the first quadratic form of the middle surface of the coating; W_0, W_0^*, density of heat sources and density of the "moments" of heat sources per unit area of the middle plane of the coating, which characterize the nonuniform distribution of sources over the coating thickness.

REFERENCES

1. Yu. M. Kolyano and Ye. P. Khomyakevich, in: Mathematical Methods and Physicomechanical Fields [in Russian], Issue 2, Kiev (1975), pp. 81-86.

2. Yu. M. Kolyano and Ye. P. Khomyakevich, in: Thermomomechanical Processes in Piecewise-Homogeneous Structural Elements [in Russian], Kiev (1978), pp. 43-50.
3. Yu. M. Kolyano, Ye. P. Khomyakevich, I. O. Goi, and M. B. Ugai, Ways for Utilization of the Afterproducts of the Production of Building Materials and Articles, Abstracts of the Reports at an All-Union Meeting, Vol. 1, Chimkent (1986), pp. 178-179.
4. Yu. M. Kolyano, Ye. P. Khomyakevich, and I. O. Goi, Inzh.-Fiz. Zh., 55, No. 4, 650-656 (1988).
5. V. Z. Vlasov, Selected Papers [in Russian], Vol. 10, Moscow (1962).
6. Ya. S. Podstrigach and Yu. M. Kolyano, Generalized Thermomechanics [in Russian], Kiev (1976).
7. Ya. S. Podstrigach and P. K. Shevchuk, Thermal Stresses in the Elements of Structures [in Russian], Issue 7, Kiev (1967).